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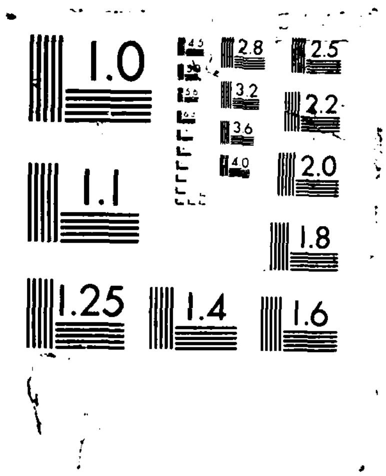
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AN EXPONENTIAL DECAY ESTIMATE FOR THE STATIONARY PERTURBATION OF POISEUILLE FLOW

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ABSTRACT

~~We prove~~ a decay estimate for the steady state incompressible Navier-Stokes equations. The estimate describes the exponential decay, in the axial direction of a semi-infinite tube, for an energy-type functional in terms of the perturbation of Poiseuille flow, provided that the Reynolds number does not exceed a critical value, for which we exhibit a lower and an upper bound. Since the motion is considered axi-symmetric we use a stream function formulation, and the results are similar to those obtained by Horgan [5], for a two-dimensional channel flow problem. For the Stokes problem our estimate for the rate of decay is a lower bound to the actual rate of decay which is obtained from an asymptotic solution to the Stokes equations.

AMS(MOS) Subject Classifications: 35Q10, 35A25

Key Words: Navier-Stokes, axi-symmetric, Poiseuille flow, decay estimate, stream function, Reynolds number

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Gerardo A. Ache*

1. Introduction

In this paper we consider the steady state Navier-Stokes equations in a semi-infinite pipe where an entrance profile is specified at inflow and the regularity condition that the base flow becomes Poiseuille flow is specified downstream at infinity. Our aim is to establish a spatial decay estimate, in the axial direction, analogous to the estimate expressing Saint Venant's principle in elasticity theory. Early results regarding the mathematical formulation and proof of decay estimates concerning Saint Venant's principle can be found in the works of Toupin [14] and Knowles [11] for linear elasticity theory in a finite cylinder. Later this type of result was extended to the case of an incompressible viscous flow motion, by Horgan and Wheeler [8] in a finite cylinder and by Horgan [5] for two dimensional incompressible viscous motion in a semi-infinite channel. Horgan and Payne [7] have established a decay estimate for second-order quasilinear partial differential equations in a semi-infinite strip.

The purpose of this work is to obtain a result analogous to those mentioned above for axi-symmetric flow in a semi-infinite pipe, i.e., we prove that an energy functional involving the stationary perturbation from Poiseuille flow decays exponentially fast in the

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axial direction, provided that the Reynolds number of the base flow does not exceed a certain quantity R_c (the critical Reynolds number) for which we exhibit a lower and an upper bound. This estimate is similar to the one presented by Horgan in [5] and it is proved using a stream function formulation which is possible since the fluid motion is considered symmetric about the axis. In [8], Horgan and Wheeler establish a decay estimate for the three-dimensional Navier-Stokes equation in a finite pipe of arbitrary cross section where Poiseuille flow was specified at the outflow boundary. In both results [5] and [8], there is a restriction in the Reynolds number for which the estimate is valid. For example, in [5] the estimate is valid for Reynolds numbers that not exceed the value of approximately 11.8, while in [8] it is valid for Reynolds numbers less than approximately 4.8. These bounds for the Reynolds numbers actually represent a lower bound for R_c. In this paper we have obtained a lower bound for R_c with a value of approximately 9.08, also we have obtained an upper bound for R_c with a value of 282.6. This last number arise from the calculation of the supremum of the quotient of two integral expressions and it plays an important role in the study of existence of solution for the incompressible Navier-Stokes equations in axi-symmetric pipes [3], and in the study of stability of Poiseuille flow [10], [12].

Similar to [5] and [8], in order to prove the decay estimate we need to establish some integral inequalities, for which we provide bounds as sharp as possible.

In section 6 we compute the rate of decay for the Stokes problem, for the stationary perturbation of Poiseuille flow, from an asymptotic solution to the Stokes equations, with a value of approximately 4.47. The rate of decay estimated in this paper has a value of

approximately 1.79, i.e. the rate of decay estimated in this paper is a lower bound for the actual rate of decay.

2. The Energy Estimate

Given a semi-infinite tube $R_0 = C_0 \times (0, \infty)$ with circular cross section C_0 of radius 1.0, we consider the steady state Navier- Stokes equations in cylindrical coordinates (r, θ, z) for a viscous incompressible fluid motion which is symmetric about the z-axis. These equations may be written as

$$u_1 \frac{\partial u_1}{\partial r} + u_2 \frac{\partial u_1}{\partial z} + \frac{\partial p}{\partial r} = \nu(\nabla^2 u_1 - u_1/r^2) \quad , \tag{2.1a}$$

$$u_1 \frac{\partial u_2}{\partial r} + u_2 \frac{\partial u_2}{\partial z} + \frac{\partial p}{\partial z} = \nu \nabla^2 u_2 \quad , \tag{2.1b}$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r u_1) + \frac{\partial u_2}{\partial z} = 0 \quad , \tag{2.1c}$$

where $\mathbf{u} = (u_1, 0, u_2)$ is the velocity field and p is the pressure. Here

$\nabla^2 \equiv 1/r \ \partial/\partial r(r\partial/\partial r) + \partial^2/\partial z^2$ and ν is the kinematic viscosity.

Associated with these equations we consider the following boundary conditions. At the wall of the tube

$$u_1(1, z) = u_2(1, z) = 0 \quad , \tag{2.1d}$$

and at the center $\mathbf{u}(0,z)$ is finite for all $z \geq 0$. We also prescribe an entrance profile $\mathbf{f} = (f_1, f_2)$ such that

$$u_1(r, 0) = f_1(r) \ \text{ and } \ u_2(r, 0) = f_2(r) \ , \ \text{ with } \ f_1(1) = f_2(1) = 0. \tag{2.1e}$$

Finally the form of the domain leads one to seek a solution $(\mathbf{u}, p)$ of (2.1) such that,

$$\mathbf{u} \to \mathbf{u}^{\infty} \text{ uniformly in } r \text{ as } z \to \infty \quad , \tag{2.1f}$$

where $\mathbf{u}^{\infty}(r,z) = (0, 0, \hat{u}(r))$ is the Poiseuille velocity field, i.e. $(\mathbf{u}^{\infty}, p^{\infty})$ is a solution of the steady state Navier-Stokes equations in an infinite pipe of cross section C_0 , and for which

$$\int_{C_0} \hat{u} dA = Q > 0 \quad . \tag{2.2}$$

Therefore $\hat{u}$ is fully described by

$$\hat{u}(r) = \frac{\bar{P}}{4\nu}(1 - r^2) = \frac{2Q}{\pi}(1 - r^2) \quad . \tag{2.3}$$

The pressure p^{∞} is such that its gradient is given by $\nabla p^{\infty} = (0, 0, -\bar{P})$, with $\bar{P}$ a positive constant.

We define a perturbation of the Poiseuille flow velocity field and pressure by

$$w_1 = u_1 \quad , \quad w_2 = u_2 - \hat{u} \quad , \quad q = p - p^{\infty} \quad . \tag{2.4}$$

Then the pair $(\mathbf{w}, q) = ((w_1, w_2), q)$ satisfies the following equations, (perturbation of Poiseuille flow equations)

$$\hat{u}\frac{\partial w_1}{\partial z} + w_1\frac{\partial w_1}{\partial r} + w_2\frac{\partial w_1}{\partial z} + \frac{\partial q}{\partial r} = \nu(\nabla^2 w_1 - \frac{w_1}{r^2}) \quad , \tag{2.5a}$$

$$\hat{u}\frac{\partial w_2}{\partial z} + \frac{d\hat{u}}{dr}w_1 + w_1\frac{\partial w_2}{\partial r} + w_2\frac{\partial w_2}{\partial z} + \frac{\partial q}{\partial z} = \nu\nabla^2 w_2 \quad , \tag{2.5b}$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(r w_1)\right) + \frac{\partial w_2}{\partial z} = 0 \quad , \tag{2.5c}$$

and boundary conditions

$$w_1(1, z) = w_2(1, z) = 0 \ . \tag{2.5d}$$

$$w_1(r,0) = f_1(r)\,,\; w_2(r,0) = f_2(r) - \hat{u}(r)\ . \tag{2.5e}$$

The condition $(2.1f)$ becomes

$$\mathbf{w} \to 0 \text{ uniformly in } r \text{ as } z \to \infty\ . \tag{2.5f}$$

An integration of (2.5c) over the semi-infinite tube R_0 gives us

$$2\pi \int_{C_0} w_2 dA = 0 \quad , \tag{2.6}$$

where we have applied the divergence theorem, so the entry profile $f_2(r)$ must satisfy

$$2\pi \int_0^1 f_2(r) r dr = Q \quad . \tag{2.7}$$

We now proceed to describe our decay estimate.

Let R_s be a sub-region of R_0 defined by $R_s = C_0 \times (s, \infty)$, similar to [5] the type of decay estimate we consider involves the following functional, defined for $0 \le s \le \infty$,

$$E(s) = \int\int_{R_s} \nabla \mathbf{w} : \nabla \mathbf{w} dV :=$$

$$2\pi \int_s^\infty \int_0^1 \Big\{ (\frac{\partial w_1}{\partial r})^2 + (\frac{\partial w_1}{\partial z})^2 + (\frac{\partial w_2}{\partial r})^2 + (\frac{\partial w_2}{\partial z})^2 - \frac{w_r^2}{r^2} \Big\} r dr dz\ . \tag{2.8}$$

Our decay estimate is stated in the following theorem, which will be proved in section 4.

Theorem 2.1 Let $(\mathbf{u}, p)$ be a classical solution for equations (2.1a) to (2.1f), with f_2 satisfying (2.7). Let $(\mathbf{w}, q)$ be the associated perturbation of Poiseuille flow as defined in (2.5) , and satisfying the following conditions

$$E(0) < \infty\ , \tag{2.9a}$$

$$\int_{R_s}^\infty E(\varsigma) d\varsigma < \infty \text{ for all } s \ge 0\ . \tag{2.9b}$$

$$w_2 \text{ is bounded ,} \tag{2.9c}$$

$$\frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z}, \frac{\partial^2 w_1}{\partial z^2} \text{ are bounded uniformly in } r \text{ as } z \to \infty . \tag{2.9d}$$

Then there exist positive constants K, α, ν_0 such that

$$E(s) \leq KE(0)\exp(-\alpha s) \quad \text{for} \quad \nu > \nu_0, \; s \geq 0. \tag{2.10}$$

We will prove this theorem using a stream function formulation. The proof parallels the result of Horgan [5], which corresponds to the case of two dimensional plane flow. The stream function formulation is discussed in the next section.

3. Stream Function Formulation

The main difficulty in establishing energy decay estimates for solutions of the Navier-Stokes equations consists of eliminating the perturbed pressure q from the estimates. In [8], Horgan and Wheeler found a device useful for this purpose. They consider the steady state Navier-Stokes equations in a finite pipe, where Poiseuille flow is specified as an outflow boundary condition, then it is possible to eliminate the perturbed pressure by the use of an auxiliary function. However their approach is not applicable to the infinite region we are considering here.

When an incompressible flow is two-dimensional or axi-symmetric one can eliminate the pressure by introducing a stream function. It is then possible to transform the perturbation of Poiseuille flow equations into a single fourth-order equation from which the estimates follows. This approach has been considered by Horgan [5], for the case of two-dimensional plane flow. Here we present a similar result for the axi-symmetric case.

We start by defining the following stream function ψ ,

$$w_1 = -\frac{1}{r}\frac{\partial\psi}{\partial z} \quad , \quad w_2 = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{3.1}$$

By use of this definition the incompressibility condition (2.5c) is automatically satisfied. Then substituting the vector field $\mathbf{w}$ by (3.1) in equations (2.5a) and (2.5b) it is possible to eliminate the perturbed pressure q in such a way that we obtain the following fourth-order equation,

$$\frac{1}{r}\hat{u}(r)\frac{\partial J^2\psi}{\partial z} + \frac{1}{r^2}\left\{\frac{\partial\psi}{\partial r}\frac{\partial J^2\psi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial J^2\psi}{\partial r}\right\} + \frac{2}{r^3}\frac{\partial\psi}{\partial z}J^2\psi = \frac{\nu}{r}J^4\psi \ , \tag{3.2a}$$

where

$$J^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \quad . \tag{3.2b}$$

We may integrate ψ so that

$$\psi(r,z) = \int_0^r w_2(y,z)y\,dy \quad , \tag{3.2c}$$

therefore, by integrating (2.5c) over R_z and applying the divergence theorem we have that $2\pi\int_0^1 w_2(r,z)r\,dr = 0$, i.e. ψ satisfies the following boundary conditions,

$$\psi(0,z) = \psi(1,z) = 0 \quad , \quad \frac{\partial}{\partial r}\psi(0,z) = \frac{\partial\psi}{\partial r}(1,z) = 0 \quad , \tag{3.2d}$$

$$\frac{\partial\psi}{\partial z}(r,0) = -rf_1(r) \equiv F_1(r) \quad , \quad \psi(r,0) = \int_0^r (f_2(y) - \hat{u}(y))y\,dy \equiv F_2(r). \tag{3.2e}$$

Condition (2.5f) becomes

$$\left(\psi, \frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial r}\right) \to (0,0,0) \quad \text{uniformly in } r \text{ as } z \to \infty. \tag{3.2f}$$

From (3.2c) and assumption (2.9c) it follows that

$$\frac{\psi}{r^2} = O(1) \quad , \tag{3.2g}$$

from (2.9d) we have that

$$\frac{1}{r}\left(\frac{\partial^2\psi}{\partial r\partial z}, \frac{\partial^2\psi}{\partial z^2}, \frac{\partial^3\psi}{\partial z^3}\right) \text{ are bounded uniformly in } r \text{ as } z \to \infty \, . \tag{3.2h}$$

Finally the energy functional (2.8) in terms of ψ is ,

$$E(s) = 2\pi \int_s^\infty \int_0^1 \Big\{ (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial z}))^2 + (\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\psi}{\partial z}))^2 - (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}))^2 +$$

$$(\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\psi}{\partial r}))^2 + (\frac{1}{r^2}\frac{\partial\psi}{\partial z}))^2 \Big\} r dr dz \tag{3.3}$$

We now proceed to establish some inequalities which are useful in proving Theorem 2.1.

Lemma 3.1

Let $f \in C^1[(0,1)]$ with $f(1) = 0$, then

$$\int_0^1 f^2 r dr \le \mu_1^{-2} \int_0^1 (\frac{d}{dr}f)^2 r dr \quad . \tag{3.4a}$$

where μ_1 is the first zero of the Bessel function of first kind and order zero i.e., $\mu_1 \doteq 2.4048$.

Proof

The function f can be expanded as $f(r) = \sum_{n=1}^\infty A_n J_0(\mu_n r)$, where J_0 denotes the Bessel function of first kind and order zero and the μ_n's satisfy $J_0(\mu_n) = 0$. For the functions $J_0(\mu_n r)$ we have the following two orthogonality relations,

$$\int_0^1 J_0(\mu_n r) J_0(\mu_m r) r dr = \int_0^1 \frac{d}{dr}J_0(\mu_n r)\frac{d}{dr}J_0(\mu_m r) r dr = 0 \quad (m \neq n).$$

By the differential equation defining $J_0(r)$ we have that

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}J_0(\mu_n r)\right) = \mu_n^2 J_0(\mu_n r).$$

Then multiplying this last expression by $rJ_0(\mu_n r)$ and integrating by parts gives,

$$\mu_n^2 \int_0^1 J_0^2(\mu_n r) r dr = \mu_n^2 \int_0^1 \left(J_0'(\mu_n r)\right)^2 r dr \ ,$$

where the prime on J_0 indicates the derivative.

Therefore ,

$$\int_0^1 f^2 r dr = \sum_{n=1}^{\infty} A_n^2 \int_0^1 J_0^2(\mu_n r) r dr = \sum_{n=1}^{\infty} \frac{A_n^2}{\mu_n^2} \int_0^1 \left(\frac{d}{dr}J_0(\mu_n r)\right)^2 r dr$$

$$\leq \mu_1^{-2} \int_0^1 \left(\frac{d}{dr}f\right)^2 r dr \qquad ,$$

where the last inequality follows since μ_1 is the smallest zero of $J_0(r)$.

Lemma 3.2

Let $f \in C^1[(0,1)]$ with $f(1) = 0$, then we have

$$\int_0^1 f^4 r^2 dr \leq \sigma_1 \left(\int_0^1 \left(\frac{d}{dr}f\right)^2 r dr\right)^2 \ , \tag{3.4b}$$

where $\sigma_1 = \frac{4}{3}\mu_1^{-2}$.

Proof

We have that

$$f^2(r) = -2\int_r^1 f(y)\frac{df}{dy}dy \leq 2\left(\int_r^1 f^2(y)dy\right)^{1/2}\left(\int_r^1 \left(\frac{d}{dt}f(t)\right)^2 dt\right)^{1/2} \ .$$

so

$$\int_0^1 f^4(r) r^2 dr \leq 4\int_0^1 \int_r^1 \int_r^1 r^2 f^2(y)\left(\frac{d}{dt}f(t)\right)^2 dy\,dt\,dr \ .$$

We may then interchange the integration in t and r , obtaining

$$\int_0^1 f^4(r) r^2 dr \le 4 \int_0^1 \left(\frac{d}{dt} f(t)\right)^2 \int_0^t \int_r^1 r^2 f^2(y) dy dr dt \quad .$$

Interchanging the integration in y and r we have that

$$\int_0^1 f^4(r) r^2 dr \le 4 \int_0^1 \left(\frac{d}{dt} f(t)\right)^2 \int_0^1 f^2(y) \int_0^{\min(y,t)} r^2 dr dy dt =$$

$$4 \int_0^1 \left(\frac{d}{dt} f(t)\right)^2 \int_0^1 f^2(y) \frac{1}{3} \min(y,t)^3 dy dt \le \frac{4}{3} \int_0^1 \left(\frac{d}{dr} f(r)\right)^2 r dr \int_0^1 f^2(r) r dr .$$

An application of (3.4a) gives the desired result.

Lemma 3.3

Let $f'/r \in C^1[(0,1)]$ with $f(1) = f'(1) = 0$ and $f/r \to 0$ as $r \to 0$, then

$$\int_0^1 \frac{f^2}{r^3} dr \le \mu_1^{-2} \int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr} f\right)\right)^2 r dr \quad , \tag{3.4c}$$

and

$$\int_0^1 \frac{f^4}{r^5} dr \le \sigma_1 \left(\int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr} f\right)\right)^2 r dr\right)^2 \quad , \tag{3.4d}$$

Proof

An integration by parts gives,

$$\int_0^1 \frac{f^2}{r^3} dr = \int_0^1 \frac{f}{r^{3/2}} \left(\frac{1}{r}\frac{d}{dr} f\right) r^{\frac{1}{2}} dr \le \left(\int_0^1 \frac{f^2}{r^3} dr\right)^{1/2} \left(\int_0^1 \left(\frac{1}{r}\frac{d}{dr} f\right)^2 r dr\right)^{1/2}$$

$$\le \mu_1^{-1} \left(\int_0^1 \frac{f^2}{r^3} dr\right)^{1/2} \left(\int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr} f\right)\right)^2 r dr\right)^{1/2} \quad ,$$

where we have used Schwarz's inequality and (3.4a). Therefore

$$\int_0^1 \frac{f^2}{r^3} dr \le \mu_1^{-2} \int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr} f\right)\right)^2 r dr \quad .$$

For the second part of the lemma we apply integration by parts, Schwarz's inequality, and (3.4a), (3.4b). We have

$$\int_0^1 \frac{f^4}{r^5}dr = \int_0^1 \frac{f^3}{r^4}\frac{d}{dr}f\,dr = \int_0^1 \frac{f^2}{r^{5/2}}\left(\frac{1}{r}\frac{d}{dr}f\right)\frac{f}{r^{1/2}}dr \leq$$

$$\left(\int_0^1 \frac{f^4}{r^5}dr\right)^{1/2}\left(\int_0^1 r\left(\frac{1}{r}\frac{d}{dr}f\right)^2\frac{f^2}{r^2}dr\right)^{1/2} ,$$

hence,

$$\left(\int_0^1 \frac{f^4}{r^5}dr\right)^{1/2} \leq \left(\int_0^1 r^2\left(\frac{1}{r}\frac{d}{dr}f\right)^4 dr\right)^{1/4}\left(\int_0^1 \frac{f^4}{r^5}dr\right)^{1/4} .$$

Finally

$$\left(\int_0^1 \frac{f^4}{r^5}dr\right)^{1/4} \leq \left(\frac{4}{3}\right)^{1/4}\left(\int_0^1 \left(\frac{1}{r}\frac{d}{dr}f\right)^2 r\,dr\right)^{1/4}\left(\int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}f\right)\right)^2 r\,dr\right)^{1/4} .$$

Then

$$\int_0^1 \frac{f^4}{r^5}dr \leq \sigma_1\left(\int_0^1 \left(\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}f\right)\right)^2 r\,dr\right)^2 .$$

Now we are ready to prove our main result.

4. Proof of Theorem 2.1

To prove Theorem 2.1 we use an approach similar to that used by Horgan [5], which consists of finding an integral inequality expression which leads to the desired result (2.10).

In appendix B it is shown that $E(s)$ can be rewritten as

$$E(s) = 2\pi\int\int_{R_s}\frac{\psi}{r}J^4\psi\,dr\,dz - 2\pi\int_{C_s}\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)^2 - \frac{1}{r}\psi\frac{\partial^2\psi}{\partial z^2}\right]dr \quad . \tag{4.1}$$

and since ψ satisfies the equation (3.2a) we have from (4.1) that

$$E(s) = \frac{2\pi}{\nu}\int\int_{Rs}\psi\left\{\frac{1}{r}\hat{u}\frac{\partial}{\partial z}J^2\psi + \frac{1}{r^2}\left[\frac{\partial\psi}{\partial r}\frac{\partial J^2\psi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial J^2\psi}{\partial r}\right] + \frac{2}{r^3}\frac{\partial\psi}{\partial z}J^2\psi\right\}drdz$$

$$-2\pi\int_{Cs}\frac{\partial}{\partial z}\left[\frac{1}{r}(\frac{\partial\psi}{\partial z})^2 - \frac{1}{r}(\frac{\partial\psi}{\partial r})^2 - \frac{1}{r}\psi\frac{\partial^2\psi}{\partial z^2}\right]dr \quad . \tag{4.2}$$

Using the definition of J^2 in (3.2b) and $\hat{u}$ in (2.4) we may transform some of the terms in (4.2) as follows,

$$\frac{1}{r}\psi\hat{u}\frac{\partial J^2\psi}{\partial z} = \frac{\partial}{\partial z}(\frac{1}{r}\hat{u}\psi J^2\psi) - \frac{\partial}{\partial r}(\frac{\hat{u}}{r}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z}) - \frac{4Q}{\pi}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z} +$$

$$\frac{1}{2}\frac{\partial}{\partial z}\left\{\frac{\hat{u}}{r}\left[(\frac{\partial\psi}{\partial r})^2 - (\frac{\partial\psi}{\partial z})^2\right]\right\} \quad , \tag{4.3a}$$

and

$$\frac{1}{r^2}\left\{\psi\frac{\partial\psi}{\partial r}\frac{\partial J^2\psi}{\partial z} - \psi\frac{\partial\psi}{\partial z}\frac{\partial J^2\psi}{\partial r}\right\} + \frac{2}{r^3}\psi\frac{\partial\psi}{\partial z}J^2\psi = \frac{1}{2}\frac{\partial}{\partial r}(\frac{1}{r^2}\psi^2\frac{\partial J^2\psi}{\partial z}) -$$

$$\frac{1}{2}\frac{\partial}{\partial z}(\frac{1}{r^2}\psi^2\frac{\partial J^2\psi}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r^3}\psi^2 J^2\psi) \quad . \tag{4.3b}$$

Then substituting these two expressions (4.3a) and (4.3b) in (4.2), an application of the Green's theorem, the boundary conditions (3.2d) and the regularity conditions (3.2f) and (3.2h) transform (4.2) to

$$E(s) = 2\pi\left\{-\frac{1}{\nu}\left(int_{Cs}\left\{\frac{\hat{u}}{r}\psi J^2\psi + \frac{1}{2}\frac{\hat{u}}{r}\left[(\frac{\partial\psi}{\partial r})^2 - (\frac{\partial\psi}{\partial z})^2\right] + \frac{1}{r^2}\psi\frac{\partial\psi}{\partial r}J^2\psi\right\}dr\right.\right.$$

$$\left.\left.+\frac{4Q}{\pi}\int\int_{Rs}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z}drdz\right) - \int_{Cs}\frac{\partial}{\partial z}\left[\frac{1}{r}(\frac{\partial\psi}{\partial z})^2 + \frac{1}{r}(\frac{\partial\psi}{\partial r})^2 - \frac{1}{r}\psi\frac{\partial^2\psi}{\partial z^2}\right]dr\right\} \quad , \tag{4.4}$$

where we have applied integration by parts on the term $\frac{1}{2}\int_{Cs}\frac{1}{r^2}\psi^2\frac{\partial}{\partial r}J^2\psi dr$.

From the definition of $E(s)$ in (2.3), we observe that the derivative of $E(s)$ with respect to s is given by,

$$\frac{d}{ds}E(s) = -2\pi\int_{C_s}\Big\{\Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial v}{\partial z}\Big)\Big)^2 - \Big(\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial v}{\partial z}\Big)\Big)^2 + \Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial \psi}{\partial r}\Big)\Big)^2 +$$

$$\Big(\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial v}{\partial r}\Big)\Big)^2 - \Big(\frac{1}{r^2}\frac{\partial v}{\partial z}\Big)^2\Big\}r\,dr \quad . \tag{4.5}$$

Therefore integrating both sides of (4.4) between the limits of s and ∞, using the expression for the derivative of $E(s)$ in (4.4) we may write an identity similar to one given in [5] i.e.,

$$\frac{d}{ds}E(s) + 4\kappa^2\int_s^\infty E(\xi)d\xi = 2\pi(-I_1 - I_2 - I_3) \quad , \tag{4.6}$$

where ,

$$I_1 = \int_{C_s}\Big\{\Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial \psi}{\partial z}\Big)\Big)^2 - \Big(\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial v}{\partial z}\Big)\Big)^2 + \Big(\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial v}{\partial r}\Big)\Big)^2 - \Big(\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial \psi}{\partial r}\Big)\Big)^2$$

$$+\Big(\frac{1}{r^2}\frac{\partial \psi}{\partial z}\Big)^2\Big\}r\,dr - 4\kappa^2\int_{C_s}\Big\{\frac{1}{r}\Big(\frac{\partial \psi}{\partial z}\Big)^2 - \frac{1}{r}\Big(\frac{\partial v}{\partial r}\Big)^2 - \frac{1}{r}\psi\frac{\partial^2 \psi}{\partial z^2}\Big\}dr \ , \tag{4.7a}$$

$$I_2 = -\frac{4\kappa^2}{\nu}\int\int_{R_s}\Big\{\frac{\hat{u}}{r}vJ^2v + \frac{1}{2}\frac{\hat{u}}{r}\Big[\Big(\frac{\partial \psi}{\partial r}\Big)^2 - \Big(\frac{\partial v}{\partial z}\Big)^2\Big] - \frac{1}{r^2}v\frac{\partial v}{\partial r}J^2v\Big\}drdz \ . \tag{4.7b}$$

$$I_3 = -\frac{16Q\kappa^2}{\nu\pi}\int_s^\infty\int\int_{R_\xi}\frac{\partial \psi}{\partial r}\frac{\partial v}{\partial z}drdzd\xi \quad . \tag{4.7c}$$

and where κ is a constant to be determined.

Our aim is to give an estimate of the integral expressions I_1, I_2, and I_3, by use of Schwarz's inequality and the inequalities stated in lemmas 2.2 to 2.4.

In appendix A it is shown that for $\kappa \leq \kappa_0$, where $\kappa_0 = (\frac{\sqrt{1+\mu_1^2}-1}{2})^{1/2}$, the integral term I_1 is non-negative. Also in appendix A it is shown that the second integral expression I_2 can be bounded as

$$I_2 \leq 4\kappa M E(s) \quad , \tag{4.8a}$$

with

$$M = \frac{\kappa}{\nu}\Big[\frac{Q}{\pi} + 2\mu_1^{-2}\frac{Q}{\pi} + \sqrt{\frac{\sigma_1}{\pi}} E(0)^{\frac{1}{2}}\Big] \quad . \tag{4.8b}$$

To estimate I_3 the double integral term in $(3.7c)$ can be bounded using

$$-\int\int_{R_\xi} \frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z} dr dz \leq \frac{\gamma_\xi}{2\pi} E(\xi) \quad , \tag{4.9}$$

where ,

$$\gamma_\xi = \sup_\phi \Big(-\int\int_{R_\xi} \frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial z} dr dz \Big) \Big/ \|\phi\|^2 \quad , \tag{4.10}$$

with

$$\|\phi\| := \Big(\int\int_{R_\xi} \Big\{ (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial z}))^2 - (\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\phi}{\partial z}))^2 + (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial r}))^2 + (\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\phi}{\partial r}))^2 + (\frac{1}{r^2}\frac{\partial\phi}{\partial z})^2 \Big\} r dr \Big)^{1/2} \quad , \tag{4.11}$$

and ϕ satisfying the boundary conditions $(3.2d)$. Then we have that γ_ξ satisfies the following conditions.

$$\gamma_\xi = \gamma_0 \quad \forall \xi \geq 0 \quad , \tag{4.12a}$$

$$\gamma_\xi \leq \frac{\mu_1^{-2}}{2} \quad . \tag{4.12b}$$

$$\gamma_\xi \geq \gamma_{-\infty} \quad . \tag{4.12c}$$

where $-\infty$ as a lower subscript stands for the supremum in (4.10) when R_ξ is replaced by an infinite cylinder with same cross section as R_0 .

Condition (4.12a) is obtained by a shifting argument in ξ. (4.12b) follows using the arithmetic-geometric inequality ((A.4) appendix A, with $\varepsilon = \frac{1}{2}$) and (3.4$a$). Finally, without loss of generality it can be assumed that the supremum in the infinite cylinder is taken over C^∞ functions of compact support. Therefore we can approximate $\gamma_{-\infty}$ arbitrarily close by considering only functions with compact support. Then for any $\hat{\phi}$ of compact support in the infinite cylinder we construct a function $v_{\hat{\phi}}$ of compact support in R_0 by a translation in z i.e., $v_{\hat{\phi}}(r,z) = \hat{\phi}(r, z-k)$ (for some $k \geq 0$). Therefore the quotient of the two integral terms in (4.10), when ϕ is replaced by $v_{\hat{\phi}}$, can be made arbitrarily close to $\gamma_{-\infty}$, hence $\gamma_0 \geq \gamma_{-\infty}$.

Therefore using (4.12a) we obtain the following bound for I_3 ,

$$I_3 \leq \frac{8Q\kappa^2\gamma_0}{\nu\pi^2}\int_s^\infty E(\xi)d\xi \quad . \tag{4.13}$$

Using these bounds for I_1, I_2, I_3, the identity (4.6) is transformed into the inequality

$$\frac{d}{ds}E(s) + 4\kappa^2\delta\int_s^\infty E(\xi)d\xi \leq 4\kappa M E(s) \quad , \tag{4.14a}$$

where

$$\delta = 1 - \frac{4Q\gamma_0}{\nu\pi} \quad . \tag{4.14b}$$

Notice that δ is positive for values of the viscosity satisfying,

$$\nu \ > \ \frac{4Q\gamma_0}{\pi}. \tag{4.15}$$

In [5] and [8], Horgan and Horgan and Wheeler have shown that the inequality

$$\frac{d}{ds}E(s) + 4\kappa^2\delta\int_s^\infty E(\xi)d\xi \leq 4\kappa M E(s) \quad , \tag{4.16}$$

leads to (2.10), where α and K are given by

$$\alpha = 2\kappa\left[\sqrt{M^2+\delta} - M\right] \quad \text{and} \quad K = \frac{2\sqrt{M^2+\delta}}{(\sqrt{M^2+\delta} - M)} \quad , \tag{4.17}$$

respectively and M and δ given by (4.8b) and (4.14b) respectively (see, e.g [5] pp. 371-372). Therefore by taking $\nu_0 = 4Q\gamma_0\ \pi$, we have completed the proof of Theorem 2.1.

Notice that using (4.12b) we can obtain the following upper bound for ν_0 i.e., $\nu_0 \le \frac{2Q}{\pi\mu_1^2}$. On the other hand using (4.12c) we obtain the lower bound. $\nu_0 \ge 4Q\gamma_{-\infty}/\pi$.

The value $\gamma_{-\infty}$ was computed in 1907 by Orr (see [12], pp. 157-159) and by Amick (1978, [4], pp. 117-119). They computed the value for $\gamma_{-\infty}$ of approximately 1/359.82 . If we define a Reynolds number by $R = \frac{Q}{\nu}$, then the restriction (4.15) in term of Reynolds number becomes,

$$R \ < \ R_c = \frac{\pi}{4\gamma_0} \quad .$$

Then, on using the upper and lower bounds for γ_0, R_c lies in the interval [9.084, 282.6] .

This value of $\gamma_{-\infty}$ plays an important role in the study of existence of solutions for the Navier-Stokes equations in axi-symmetric pipes [3], and in the study of stability of Poiseuille flow [10], [12].

5. Estimate for E(0)

The estimate (2.10) depends on the value $E(0)$, for which it may be convenient to provide an upper bound. As pointed out in [5] and [8], the estimation for $E(0)$ can be given in term of a functional involving an appropriate extension of the solution for the Stokes problem in R_0. Such an estimation will introduce another restriction to the viscosity or Reynolds number.

The Stokes problem, in the stream function formulation, can be regarded as equation (3.2a) for which the viscosity tends to infinity i.e., if η represents the stream function for the Stokes problem then we have that η satisfies,

$$J^4\eta = 0 \quad \text{in} \quad R_0, \tag{5.1a}$$

with boundary conditions

$$\eta\,(r,z) = \frac{\partial}{\partial r}\eta(r,z) = 0 \qquad \text{at} \quad r = 0,1 \quad , \tag{5.1b}$$

$$\eta\,(r,0) = F_2(r) \quad , \quad \frac{\partial}{\partial z}\eta(r,0) = F_1(r) \quad , \tag{5.1c}$$

and the regularity conditions

$$(\eta, \frac{\partial\eta}{\partial r}, \frac{\partial\eta}{\partial z}) \to (0,0,0) \qquad \text{uniformly in } r \text{ as } z \to \infty \tag{5.1d}$$

$$\frac{\eta}{r^2} = O(1) \quad , \tag{5.1e}$$

and

$$\frac{1}{r}(\frac{\partial^2\eta}{\partial r\partial z}, \frac{\partial^2\eta}{\partial z^2}, \frac{\partial^3\eta}{\partial z^3}) \text{ are bounded uniformly in } r \text{ as } z \to \infty \quad . \tag{5.1f}$$

We define an energy functional, $E_\infty(s)$, associated with η by the formula (3.3) with ψ replaced by η where the infinity as lower subscript refers for the infinite viscosity. From (4.8b), (4.14b) and (4.17) we have the following estimate for $E_\infty(s)$,

$$E_\infty(s) \le 2E_\infty(0)\exp(-2\kappa s) \quad , \quad s \ge 0. \tag{5.2}$$

Also it is possible to give the following bound for $E(0)$ in terms of $E_\infty(0)$, (the derivation is provided in appendix B).

$$E\,(0) \le \frac{E_\infty(0)}{(\frac{1}{2} - \frac{\pounds}{\nu^2})} \quad , \tag{5.3}$$

with ϱ a constant given by

$$\varrho = \left(\frac{2\sqrt{2}Q}{\pi\mu_1}\right)^2 + 2\frac{E_\infty(0)}{\pi\mu_1^2} \quad . \tag{5.4}$$

In order that (5.4) be well defined we need the viscosity be sufficiently large so that

$$\nu \quad > \quad (2\varrho)^{1/2}.$$

Using a modification of the argument in [13], (see appendix C), we can prove that for the Stokes problem (5.1) we have the following minimum principle,

$$E_\infty(0) = 2\pi \min_{\phi} \mathcal{B}_0(\phi, \phi) \quad , \tag{5.5}$$

(the bilinear functional $\mathcal{B}_0$ is defined in (B.1) appendix B), with ϕ satisfying conditions (5.1b) to (5.1e). Since for any function $\tilde{\phi}$ satisfying conditions (5.1b) to (5.1e) we have that $E_\infty(0) \leq 2\pi\mathcal{B}_0(\tilde{\phi}, \tilde{\phi})$ then upper bounds for $E_\infty(0)$, and consequently for $E(0)$, may be obtained on constructing a suitable auxiliary function $\tilde{\phi}$.

6. The Rate of Decay for the Stoke Problem

In this section we compute the rate of decay associated with the Stokes equations. These equations are obtained by setting the viscosity equal to infinity (or the Reynolds number equal to zero) in equations (2.1a) to (2.1c).

Seeking for asymptotic solutions from the perturbation of Poiseuille flow equations (2.5a)-(2.5c) in the form,

$$w_1(r, z) = W_1(r)\exp(-\lambda z) \ , \tag{6.1a}$$

$$w_2(r, z) = W_2(r)\exp(-\lambda z) \ , \tag{6.1b}$$

and

$$q(r,z) = \bar{q}(r)\exp(-\lambda z) + \tilde{C} \ , \tag{6.1c}$$

with $\tilde{C}$ an arbitrary constant, yields to the following system of ordinary differential equations

$$\frac{dW_1}{dr} = \lambda W_2 - \frac{W_1}{r} \ , \tag{6.2a}$$

$$\frac{d^2W_2}{dr^2} = -\frac{1}{r}\frac{dW_2}{dr} - \lambda^2 W_2 - \lambda\bar{q} \ , \tag{6.2b}$$

$$\frac{d\bar{q}}{dr} = \lambda\frac{dW_2}{dr} + \lambda^2 W_1 \ . \tag{6.2c}$$

with boundary conditions, at the wall,

$$W_1(1) = W_2(1) = 0 \ . \tag{6.2d}$$

In order to compute a solution to this system, we may impose the following symmetric condition at the axis

$$W_1(0) = \frac{dW_2}{dr}(0) = 0 \ . \tag{6.2e}$$

To find an explicit solution to this system differentiate (6.2c) with respect to r, multiply (6.2a) by λ^2 and (6.2b) by λ, then add the resulting expressions and using (6.2c) we obtain the following differential equation in $\bar{q}$,

$$\frac{d^2\bar{q}}{dr^2} + \frac{1}{r}\frac{d\bar{q}}{dr} + \lambda^2\bar{q} = 0 \ . \tag{6.3}$$

This equation has the particular solution,

$$\bar{q}(r) = J_0(\lambda r) \ . \tag{6.4}$$

where J_0 is the Bessel function of first kind and order zero. This is the only solution, up to a multiplicative constant, which is finite when r equal to zero. Substituting $\bar{q}(r)$ in $(6.2b)$ we obtain the following differential equation in W_2,

$$\frac{d^2W_2}{dr^2} + \frac{1}{r}\frac{dW_2}{dr} + \lambda^2 W_2 = -\lambda J_0(\lambda r) \ , \tag{6.5a}$$

with boundary conditions,

$$\frac{dW_2}{dr}(0) = W_2(1) = 0 \ . \tag{6.5b}$$

This two-point boundary value problem has the following solution,

$$W_2(r) = \bar{b} J_0(\lambda r) - \frac{1}{2} r J_1(\lambda r) \ , \tag{6.6}$$

where $\bar{b}$ is a constant to be determined by $W_2(1) = 0$. Using $(6.2a)$ then W_1 is given as.

$$W_1(r) = \frac{1}{\lambda}\left[\frac{\lambda}{2} r J_0(\lambda r) - (1 - \lambda\bar{b}) J_1(\lambda r)\right] \ . \tag{6.7}$$

Since the solutions (6.6) and (6.7) need to satisfy the boundary conditions $(6.2d)$ we have that,

$$\lambda W_1(1) = \frac{\lambda}{2} J_0(\lambda) - (1 - \lambda\bar{b}) J_1(\lambda) = 0 \quad . \tag{6.8a}$$

and

$$W_2(1) = \bar{b} J_0(\lambda) - \frac{1}{2} J_1(\lambda) = 0 \ . \tag{6.8b}$$

By eliminating $\bar{b}$ from these two equations we obtain,

$$\frac{1 \pm \sqrt{1 - \lambda^2}}{2\lambda} J_0(\lambda) - \frac{1}{2} J_1(\lambda) = 0 \ . \tag{6.9}$$

To solve this equation we observe that (6.9) can be transformed, after some algebraic manipulations, into the following non-linear equation

$$J_1(\lambda) J_0(\lambda) - \frac{\lambda}{2}\left(J_1^2(\lambda) - J_0^2(\lambda)\right) = 0 \ . \tag{6.10}$$

Then the stream function $\eta(r,z)$ associated to the equation (5.1*a*), has an asymptotic behavior of the form

$$\eta(r,z) \sim \varphi(r)\exp(-\alpha z) , \tag{6.11}$$

where α is the real part of λ satisfying (6.10). The nonlinear equation (6.10) can be numerically solved using Newton method [1], the rate of decay α is given by the real part of the non-zero solution of (6.10) with smallest modulus. We compute the value for α of approximately 4.4663. In the estimate of section 4 the rate of decay is given by (4.17), i.e., for the Stokes problem, the estimated rate of decay has a value of $\alpha = 2\kappa_0$ approximately 1.7906 which differs from the computed rate from (6.9) by a factor of approximately 2.5. That is, the estimated rate of decay given by (4.17) is a lower bound for the actual rate of decay.

7. Conclusion

An energy estimate of the form (2.10) has been presented for the Navier-Stokes equations in a semi-infinite pipe for a motion which is axi-symmetric. The main result is stated in Theorem 2.1. The estimate depends on the values $E(0)$ and κ, where $E(0)$ is assumed to be finite, this assumption is typical in the study of existence for solutions of the Navier-*Stokes equations* [3]. *The value of* κ_0 *is given by* κ *approximately* .8956. For the Stokes problem κ represents a lower bound for the actual decay rate which is given by the real part of the solution, with smallest modulus, of the nonlinear equation (6.10). Finally for this proof of the exponential decay estimate it is required that the Reynolds number does not exceed a critical value R_c which lies in the interval [9.08,282.6].

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APPENDIX A

In this appendix we will derive bounds for I_1, I_2 which are define by (4.7a) and (4.7b).

—Non-negativity of I_1 :

From the definition of I_1 by applying lemma 3.1 we have.

$$-\int_{Cs}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)^2 r\,dr \geq -\mu_1^{-2}\int_{Cs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 r\,dr$$

and

$$-\int_{Cs}\left(\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right)^2 r\,dr \geq -\mu_1^{-2}\int_{Cs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 r\,dr.$$

Also,

$$4\kappa^2\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\psi \geq -\frac{1}{r}\left(\frac{\partial^2\psi}{\partial z^2}\right)^2 - 4\kappa^4\frac{\psi^2}{r^3} \quad .$$

so

$$4\kappa^2\int_{Cs}\frac{1}{r}\frac{\partial^2\psi}{\partial z^2}\psi\,dr \geq -\int_{Cs}\left(\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 r\,dr - 4\kappa^4\mu_1^{-2}\int_{Cs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 r\,dr \quad .$$

where we have applied (3.4c) from lemma 3.3.

Therefore,

$$I_1 \quad \geq \quad (1-4\kappa^2\mu_1^{-2})\int_{Cs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 r\,dr \quad -$$

$$(1-4\kappa^2\mu_1^{-2}-4\kappa^4\mu_1^{-2})\int_{Cs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 r\,dr \quad . \qquad (A.1a)$$

Since we want $I_1 \geq 0$, by setting $1-4\kappa^2\mu_1^{-2}-4\kappa^4\mu_1^{-2} = 0$ we may pick $\kappa \leq \left(\frac{\sqrt{1+\mu_1^2}-1}{2}\right)^{1/2} \doteq .895$ and with this choice of κ we have $I_1 \geq 0$.

To bound the integral term I_2 , following [5], we may split I_2 as

$$\frac{\nu I_2}{4\kappa^2} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \quad , \quad with \qquad (A.2)$$

$$\mathcal{L}_1 = \int\int_{Rs} \frac{\hat{u}}{2r}\left[\left(\frac{\partial\psi}{\partial z}\right)^2 - \left(\frac{\partial\psi}{\partial r}\right)^2\right] dr dz \quad , \tag{A.2a}$$

$$\mathcal{L}_2 = \int\int_{Rs} -\frac{\hat{u}}{r}\psi J^2\psi dr dz \quad , \tag{A.2b}$$

$$\mathcal{L}_3 = \int\int_{Rs} -\frac{1}{r^2}\frac{\partial\psi}{\partial r}\psi J^2\psi dr dz \quad . \tag{A.2c}$$

By taking $f = \partial\psi/\partial z$ in $(2.4a)$ we have that

$$\mathcal{L}_1 \leq \int\int_{Rs} \frac{\hat{u}}{2}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)^2 r dr dz \leq \mu_1^{-2}\frac{Q}{\pi}\int\int_{Rs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 r dr dz \quad , \tag{A.3}$$

where $\max_{[0,1]} \hat{u} = \frac{2Q}{\pi}$. We can bound $\mathcal{L}_2$ by

$$\mathcal{L}_2 \leq \frac{2Q}{\pi}\int\int_{Rs}\left(\varepsilon\frac{\psi^2}{r} - \frac{1}{4\varepsilon r}(J^2\psi)^2\right) dr dz \quad ,$$

where we have used the inequality

$$a\,b \leq \varepsilon a^2 - \frac{b^2}{4\varepsilon} \quad . \tag{A.4}$$

Moreover.

$$\int\int_{Rs}\frac{1}{r}(J^2\psi)^2 dr dz = \int\int_{Rs}\frac{1}{r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{\partial^2\psi}{\partial z^2}\right)^2 dr dz \leq$$
$$2\int\int_{Rs}\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 - \left(\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2\right\} r dr dz \ . \tag{A.5}$$

By taking $\varepsilon - 1$ and applying $(3.4c)$ from lemma 3.3. we have the following bound for $\mathcal{L}_2$.

$$\mathcal{L}_2 \quad \leq \quad 2\mu_1^{-2}\frac{Q}{\pi}\int\int_{Rs}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 r dr dz \quad -$$

$$\frac{Q}{\pi}\int\int_{Rs}\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 - \left(\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2\right\} r dr dz \ . \tag{A.6}$$

To bound $\mathcal{L}_3$ we have that an application of Schwarz's inequality give us

$$\int\int_{Rs} \frac{\psi}{r} J^2\psi \frac{1}{r}\frac{\partial\psi}{\partial r} drdz \leq$$

$$\Big(\int\int_{Rs} \frac{\psi^2}{r}(\frac{1}{r}\frac{\partial\psi}{\partial r})^2 drdz\Big)^{1/2}\Big(\int\int_{Rs} \frac{(J^2\psi)^2}{r} drdz\Big)^{1/2} , \qquad (A.7)$$

then another application of Schwarz's inequality gives

$$\mathcal{L}_3 \leq \Big(\int\int_{Rs} \frac{\psi^4}{r^4} drdz\Big)^{1/4}\Big(\int\int_{Rs} r^2(\frac{1}{r}\frac{\partial\psi}{\partial r})^4 drdz\Big)^{1/4} \frac{1}{\sqrt{\pi}}(E(0))^{1/2} ,$$

where we have used $(A.5)$ and the monotonicity of $E(s)$.

By applying (3.4b) with $f = \frac{1}{r}\frac{\partial\psi}{\partial r}$ we have

$$\int\int_{Rs} r^2(\frac{1}{r}\frac{\partial\psi}{\partial r})^4 drdz \leq \sigma_1\Big(\int\int_{Rs} (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}))^2 rdrdz\Big)^2 .$$

and using (3.4d) we obtain

$$\mathcal{L}_3 \leq \sqrt{\frac{\sigma_1}{\pi}} E(0)^{\frac{1}{2}} \int\int_{Rs} \Big(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r})\Big)^2 rdrdz , \qquad (A.8)$$

Combining the estimates for $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ we have the following estimate for I_2

$$I_2 \leq \frac{4\kappa M E(s)}{2\pi} , \qquad (A.9)$$

where $$M = \frac{\kappa}{\nu}\Big[\frac{Q}{\pi} + 2\mu_1^{-2}\frac{Q}{\pi} + \sqrt{\frac{\sigma_1}{\pi}} E(0)^{\frac{1}{2}}\Big] .$$

APPENDIX B

In this appendix we derive the expression (4.1) and the bound (5.4).

First we define the functional $\mathcal{B}_s$ for $s \geq 0$ by,

$$\mathcal{B}_s(\tilde{\psi},\tilde{\eta}) = \int\int_{R_s}\Big\{\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z})\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}) - \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z})\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}) +$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r})\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}) - \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r})\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}) - \frac{1}{r^4}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial\tilde{\eta}}{\partial z}\Big\}r\,dr\,dz\ . \qquad (B.1)$$

where $\tilde{\psi}$ and $\tilde{\eta}$ belong to the class $C^4(R_s)$ and satisfy the boundary conditions $(3.2d)$, and $(3.2e)$ and the regularity conditions $(3.2f)$, $(3.2g)$, and $(3.2h)$. We first prove that the functional $\mathcal{B}_s$ satisfies the following relation,

$$\mathcal{B}_s(\tilde{\psi},\tilde{\eta}) = \int\int_{R_s}\frac{1}{r}\tilde{\psi}J^4\tilde{\eta}\,dr\,dz - \int_{C_s}\Big\{\frac{2}{r}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial^2\tilde{\eta}}{\partial z^2} - \frac{2}{r}\frac{\partial\tilde{\psi}}{\partial r}\frac{\partial^2\tilde{\eta}}{\partial r\partial z} - \frac{\partial}{\partial z}(\frac{1}{r}\tilde{\psi}\frac{\partial^2\tilde{\eta}}{\partial z^2})\Big\}dr\ .$$
$$(B.2)$$

We can prove $(B.2)$ as follows. Starting with $(B.1)$ an integration by parts in r and z gives,

$$\mathcal{B}_s(\tilde{\psi},\tilde{\eta}) = -\int\int_{R_s}\Big\{\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z}\Big[\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z})) - \frac{\partial}{\partial z}(r\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}))\Big] -$$

$$\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}\Big[\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}) + \frac{\partial}{\partial z}(r\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}))\Big]\Big\}dr\,dz\ -$$

$$\int_{C_s}(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial^2\tilde{\eta}}{\partial z^2} + \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}\frac{\partial^2\tilde{\eta}}{\partial r\partial z})dr\quad - \int\int_{R_s}\frac{1}{r^3}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial\tilde{\eta}}{\partial z}dr\,dz\ .$$

Then another integration by parts gives,

$$\mathcal{B}_s(\tilde{\psi},\tilde{\eta}) \quad = \int\int_{R_s}\tilde{\psi}\Big[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}))) + \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial z}(r\frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r})))$$

$$+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}\right)\right)\right)-\frac{1}{r}\frac{\partial^4\tilde{\eta}}{\partial z^4}\Big]drdz-\int_{C_s}\frac{1}{r}\left[\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial^2\tilde{\eta}}{\partial z^2}-\frac{\partial\tilde{\psi}}{\partial r}\frac{\partial^2\tilde{\eta}}{\partial r\partial z}\right]dr$$

$$+\int_{C_s}\tilde{\psi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}\right)\right)-\frac{1}{r}\frac{\partial^3\tilde{\eta}}{\partial z^3}\right]drdz-\int\int_{R_s}\frac{1}{r^3}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial\tilde{\eta}}{\partial z}drdz\ . \qquad (B.3)$$

Using the definition of J^2 in (3.2b) the first two terms on the right hand side of $(B.3)$ can be rewritten as

$$\tilde{\psi}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}\right)\right)\right)+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial r}\right)\right)\right)\right\}=\tilde{\psi}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}J^2\tilde{\eta}\right)\ , \qquad (B.4a)$$

the next two terms can be rewritten as

$$\tilde{\psi}\left\{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}\right)\right)\right)-\frac{1}{r}\frac{\partial^4}{\partial z^4}\tilde{\eta}\right\}=\tilde{\psi}\left\{\frac{1}{r}\frac{\partial^2}{\partial z^2}J^2\tilde{\eta}+\frac{1}{r^3}\frac{\partial^2\tilde{\eta}}{\partial z^2}\right\}\ . \qquad (B.4b)$$

the last term in $(B.3)$ becomes

$$\frac{1}{r^3}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial\tilde{\eta}}{\partial z}=\frac{\partial}{\partial z}\left(\frac{1}{r^3}\tilde{\psi}\frac{\partial\tilde{\eta}}{\partial z}\right)-\frac{1}{r^3}\tilde{\psi}\frac{\partial^2\tilde{\eta}}{\partial z^2} \qquad (B.4c)$$

the two terms in the next to the last integral of $(B.3)$ can be expressed as

$$\frac{\tilde{\psi}}{r}\frac{\partial^3\tilde{\eta}}{\partial z^3}=\frac{\partial}{\partial z}\left(\frac{1}{r}\tilde{\psi}\frac{\partial^2\tilde{\eta}}{\partial z^2}\right)-\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z}\frac{\partial^2\tilde{\eta}}{\partial z^2}\ . \qquad (B.4d)$$

and

$$\frac{\tilde{\psi}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\tilde{\eta}}{\partial z}\right)\right)=\frac{\tilde{\psi}}{r^3}\frac{\partial\tilde{\eta}}{\partial z}+\frac{\partial}{\partial r}\left(\frac{\tilde{\psi}}{r}\frac{\partial^2\tilde{\eta}}{\partial r\partial z}\right)-\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}\frac{\partial^2\tilde{\eta}}{\partial r\partial z}\ . \qquad (B.4e)$$

Using the expressions $(B.4a)$ to $(B.4e)$ in $(B.3)$ and integrating by parts we obtain $(B.2)$. Since $E(s)=2\pi\mathcal{B}_s(\psi,\psi)$, where ψ is a solution to (3.2), then (4.1) follows replacing $\tilde{\psi}$ and $\tilde{\eta}$ by ψ in $(B.2)$.

In the case that $\tilde{\psi} = \psi$, $\tilde{\eta} = \eta$ and $s = 0$. where ψ and η are solutions to (3.2) and (5.1) respectively, we can prove the following identity, which is useful to furnish a bound to $E(0)$,

$$\mathcal{B}_0(\psi, \psi) = \mathcal{B}_0(\eta, \eta) + \mathcal{B}_0(\psi - \eta, \psi - \eta) \quad . \tag{B.5}$$

which follows provided that $\mathcal{B}_0(\psi, \eta) = \mathcal{B}_0(\eta, \eta)$. To prove this last identity we use that $J^4\eta = 0$ in R_0 and $\partial\psi/\partial z = \partial\eta/\partial z$, $\partial\psi/\partial r = \partial\eta/\partial r$ and $\psi = \eta$ at $z = 0$, therefore by using $(B.2)$ we have that

$$\mathcal{B}_0(\psi, \eta) = \int\int_{R_0} \frac{1}{r}\eta J^4\eta drdz - \int_{C_0} \left\{ \frac{2}{r}\left[\frac{\partial\eta}{\partial z}\frac{\partial^2\eta}{\partial z^2} + \frac{\partial\eta}{\partial r}\frac{\partial^2\eta}{\partial r\partial z}\right] - \frac{\partial}{\partial z}(\frac{1}{r}\eta\frac{\partial^2\eta}{\partial z^2})\right.$$

$$= \mathcal{B}_0(\eta, \eta) \ .$$

Our goal is to give a bound to $E(0)$ in terms of $E_\infty(0)$, to do that we proceed as follows. Since η, ψ and their first derivatives have the same value at the inflow boundary then, by using $(B.2)$, we have

$$\mathcal{B}_0(\psi - \eta, \psi - \eta) = \int\int_{R_0} (\frac{\psi - \eta}{r})J^4(\psi - \eta)drdz$$

$$= \frac{1}{\nu}\int\int_{R_0} (\psi - \eta)\left\{\frac{\hat{u}}{r}\frac{\partial}{\partial z}J^2\psi - \frac{1}{r^2}\left[\frac{\partial\psi}{\partial r}\frac{\partial J^2\psi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}J^2\psi\right] - \frac{2}{r^3}\frac{\partial\psi}{\partial z}J^2\psi\right\}drdz. \tag{B.6}$$

where we have used that ψ satisfies equation $(3.2a)$ and $J^4\eta = 0$. Using the values of ψ and η at the boundary and the regularity conditions we can rewrite $\mathcal{B}_0(\psi - \eta, \psi - \eta)$ as

$$\mathcal{B}_0(\psi - \eta, \psi - \eta) = \frac{1}{\nu}(\mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_3) \quad . \tag{B.7}$$

where

$$C_1 = \int\int_{R_0} (\psi - \eta) \frac{\hat{u}}{r} \frac{\partial J^2 \psi}{\partial z} drdz = -\int\int_{R_0} \frac{\hat{u}}{r} \frac{\partial}{\partial z} (\psi - \eta) J^2 \psi drdz \quad , \qquad (B.8a)$$

$$C_2 = \int\int_{R_0} \frac{(\psi - \eta)}{r^2} \frac{\partial \psi}{\partial r} \frac{\partial J^2 \psi}{\partial z} drdz \qquad (B.8b)$$

$$= \int\int_{R_0} \left[-\frac{1}{r^2} \frac{\partial}{\partial z} (\psi - \eta) \frac{\partial \psi}{\partial r} J^2 \psi - \frac{(\psi - \eta)}{r^2} \frac{\partial^2 \psi}{\partial r \partial z} J^2 \psi \right] drdz \quad ;$$

and

$$C_3 = \int\int_{R_0} \frac{(\psi - \eta)}{r^2} \left[-\frac{\partial \psi}{\partial z} \frac{\partial J^2 \psi}{\partial r} - \frac{2}{r^3} \frac{\partial \psi}{\partial z} J^2 \psi \right] drdz \qquad (B.8c)$$

$$= \int\int_{R_0} \left[\frac{1}{r^2} \frac{\partial}{\partial r} (\psi - \eta) \frac{\partial \psi}{\partial z} J^2 \psi - \frac{(\psi - \eta)}{r^2} \frac{\partial^2 \psi}{\partial r \partial z} J^2 \psi \right] drdz \quad .$$

We define S_1, S_2 by

$$S_1 = \int\int_{R_0} \frac{1}{r^2} \frac{\partial (\psi - \eta)}{\partial r} \frac{\partial \eta}{\partial z} J^2 \psi drdz \qquad (B.9a)$$

and

$$S_2 = -\int\int_{R_0} \frac{1}{r^2} \frac{\partial}{\partial z} (\psi - \eta) \frac{\partial \eta}{\partial r} J^2 \psi drdz \quad . \qquad (B.9b)$$

Then, by a simple algebraic manipulation, the following relation is satisfied .

$$C_2 - C_3 = S_1 - S_2 \quad .$$

On applying Schwarz's inequality twice and $(A.5)$ have that.

$$S_1 \leq$$

$$\left(\int\int_{R_0} (\frac{1}{r} \frac{\partial}{\partial r} (\psi - \eta))^4 r drdz \right)^{1/4} \left(\int\int_{R_0} (\frac{1}{r} \frac{\partial \eta}{\partial z})^4 r drdz \right)^{1/4} (2\mathcal{B}_0(\psi, \psi))^{1/2}$$

and

$$S_2 \le$$

$$\left(\int\int_{R_0}\left(\frac{1}{r}\frac{\partial}{\partial z}(\psi-\eta)\right)^4 r\,dr\,dz\right)^{1/4}\left(\int\int_{R_0}\left(\frac{1}{r}\frac{\partial\eta}{\partial r}\right)^4 r\,dr\,dz\right)^{1/4}(2\mathcal{B}_0(\psi,\psi))^{1/2}.$$

To furnish the bounds for S_1 and S_2 we need an additional inequality

$$\int_0^1 f^4 r\,dr \le 2\mu_1^{-2}\left(\int_0^1\left(\frac{d}{dr}f\right)^2 r\,dr\right)^2 . \qquad (B.10)$$

for $f \in C^1[(0,1)]$ with $f(1)=0$.

This inequality follows in the same manner as 3.4b in lemma 3.2 . Using inequality $(B.10)$ when $f = \frac{1}{r}\frac{\partial}{\partial r}(\psi-\eta)$, $\frac{1}{r}\frac{\partial}{\partial z}(\psi-\eta)$, $\frac{1}{r}\frac{\partial\eta}{\partial r}$, and $\frac{1}{r}\frac{\partial\eta}{\partial z}$, we get

$$S_1 + S_2 \le 2\mu_1^{-1}(\mathcal{B}_0(\psi-\eta,\psi-\eta))^{1/2}(\mathcal{B}_0(\eta,\eta))^{1/2}(\mathcal{B}_0(\psi,\psi))^{1/2} \qquad (B.11)$$

Using Schwarz's inequality and inequalities $(3.4a)$ and $(A.5)$ we get the following bound for C_1 .

$$C_1 \le \frac{2\sqrt{2}Q}{\pi}\mu_1^{-1}(\mathcal{B}_0(\psi-\eta,\psi-\eta))^{1/2}(\mathcal{B}_0(\psi,\psi))^{1/2} . \qquad (B.12)$$

Therefore using these bounds for C_1, and $C_2 + C_3$ in $(B.11)$ we have that

$$(\mathcal{B}_0(\psi-\eta,\psi-\eta))^{1/2} \le \frac{1}{\nu}\left(\frac{2\sqrt{2}Q\mu_1^{-1}}{\pi} + 2\mu_1^{-1}\mathcal{B}_0(\eta,\eta)^{\frac{1}{2}}\right)(\mathcal{B}_0(\psi,\psi))^{1/2} . \qquad (B.13)$$

or

$$\mathcal{B}_0(\psi-\eta,\psi-\eta) \le \frac{2}{\nu^2}\left[\left(\frac{2\sqrt{2}Q\mu_1^{-1}}{\pi}\right)^2 + 4\mu_1^{-2}\mathcal{B}_0(\eta,\eta)\right]\mathcal{B}_0(\psi,\psi) . \qquad (B.14)$$

Finally using $(B.3)$ we have that

$$E(0) \le \frac{E_\infty(0)}{\frac{1}{2} - \frac{1}{\nu^2}\left[\left(2\sqrt{2}Q\mu_1^{-1}/\pi\right)^2 + 2\mu_1^{-2}E_\infty(0)/\pi\right]} . \qquad (B.15)$$

where we have used $(B.5)$ and that $\mathcal{B}_0(\eta,\eta) = E_\infty(0)/2\pi$, and $\mathcal{B}_0(\psi,\psi) = E(0)/2\pi$. Then $(B.15)$ gives the desired bound (5.4) for $E(0)$ in terms of $E_\infty(0)$.

APPENDIX C

In this appendix we give a brief description of a minimun principle associated with the Stokes problem (5.1). We consider the following variational problem ,

$$I(\phi^*) = \min_{\phi} I(\phi) \quad . \qquad (C.1)$$

where $I(\phi) = \mathcal{B}_0(\phi, \phi)$. The admissible functions ϕ belong to the class $C^4(R_0)$ and satisfy boundary conditions (5.1b), (5.1c) and the regularity conditions (5.1d), (5.1e), (5.1f).

We assume that the minimun in $(C.1)$ is attained for some ϕ^*. By representing ϕ as $\phi = \varphi^* - \varepsilon\chi$ we consider the variation δI , defined by

$$\delta I = \frac{\partial I(\phi^* - \varepsilon\chi)}{\partial \varepsilon} \Big|_{\varepsilon=0} \quad . \qquad (C.2)$$

Substituting ϕ by $\phi^* - \varepsilon\chi$ in $I(\phi)$ we have that .

$$\delta I = 2\mathcal{B}_0(\varphi^*, \chi) = 2\int\int_{R_0} \frac{\chi}{r} J^4 \phi^* \, dr dz - 2\int_{C_0} \Big\{ \frac{2}{r}\frac{\partial \chi}{\partial z}\frac{\partial^2 \phi^*}{\partial r^2} -$$

$$\frac{2}{r}\frac{\partial \chi}{\partial r}\frac{\partial^2 \phi^*}{\partial r \partial z} - \frac{\partial}{\partial z}\Big(\frac{\chi}{r}\frac{\partial^2 \phi^*}{\partial z^2}\Big) - \frac{2\chi}{r^3}\frac{\partial \phi^*}{\partial z}\Big\} dr \quad .$$

where we have used $(B.2)$. Since we want that $\delta I = 0$ then

$$2\int\int_{R_0} \frac{\chi}{r} J^4 \phi^* \, dr dz = 0 \quad .$$

since χ , $\frac{\partial \chi}{\partial r}$ and $\frac{\partial \chi}{\partial z}$ vanish at $z = 0$. Since χ is arbitrary we then conclude that $J^4\phi^* = 0$. Therefore $E_\infty(0) = 2\pi I(\phi^*)$.

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